Asymptotic Capacity of Wireless Ad Hoc Networks with Realistic Links under a Honey Comb Topology

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***Abstract*— We consider the effects of Rayleigh fading and lognormal shadowing in the physical interference model for all the successful transmissions of traffic across the network. New bounds are derived for the capacity of a given random ad hoc wireless network that reflect packet drop or capture probability of the transmission links. These bounds are based on a simplified network topology termed as *honey-comb topology* under a given routing and scheduling scheme.**

***Index Terms*— Physical Interference Model, Rayleigh Fading, Throughput Capacity.**

I. Introduction

Wireless ad hoc networks have become a hub of cutting edge research due to their applications in various fields such as Internet, sensor networks and military communications. The throughput capacity of an ad hoc network is defined as the number of scheduled packets per slot successfully received by the destination.

In a seminal work [1], the authors estimate asymptotic capacity bounds of a wireless ad hoc network. This work, however, does not consider link failures, fading, delay, mobility and traffic variability. The authors in [2] consider the effects of different traffic patterns on the scalability of throughput capacity. The capacity problem of wireless ad hoc network has also been explored in [3], [4] and [5]. In [6] and [7], the authors analyze node isolation probability under the effect of fading and shadowing. In [8], the authors derive capacity for a more realistic channel which incorporates packet drop or capture probability.

In this paper, we extend the analysis in [8] and tighten the $O(\frac{1}{n})$ result and prove that capacity scales as $\Theta(\frac{1}{n})$ In [1], [8], the system model and the ensuing analysis is based on Voronoi Tessellations. In this paper, we consider an intuitive and a more simplified network model, *a hexagonal* or *honey-comb topology* that lends itself to an amenable and elegant analysis of capacity under fading. In [9] also, the authors have considered fading but with strategies to combat fading thereby achieving linear growth. In this paper, we essentially consider the model of [8] and ask the question of how channel condition affects capacity without considering any opportunistic scheme.

II. Network Model

Consider the network topology as in Figure 1. The surface S^2 of unit radius lying on a sphere is hexagonally packed with circular discs of radius ρ_n. We then construct a honey-comb like topology comprising hexagons symmetrically tiled around any of the circular disc with vertices on the centers of the tangent discs. In this paper, we use a Hexagonal topology H_n in which every hexagonal cell encloses a disc of radius $\rho_n = \sqrt{\frac{\log n}{n}}$.

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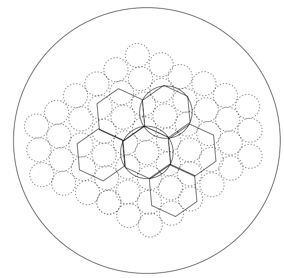

Fig. 1. Hexagonal or Honey-comb Topology. Each dotted circle has radius ρ_n while the hexagon lies in a circle of radius $2\rho_n$.

III. Routing and Scheduling Schemes

Let n be the number of nodes and $2\rho_n$ be the edge length of a hexagonal cell. It can be argued that each node contains at least one node and the number of nodes in each cell is upper bounded by $M = \frac{12\sqrt{3}\log n}{\pi}$ (refer to Appendix) with high probability (*whp*), i.e., the probability tends to one for large n.

A. Routing Scheme

We reuse the multihop routing scheme of [8] and choose the transmission range as $r_n = 8\rho_n$, which is twice the combined diameter of any two adjacent cells. Hence, the traffic generated at a source is forwarded in hops from one cell to another where all the hops occur only between nodes in neighboring cells. If L_i is the distance between source and destination, it can easily proved (as in [8]) that the number of hops H_i scales as

$$\frac{L_i}{8\rho_n} \leq H_i \leq \frac{4L_i}{3\sqrt{3}\rho_n}. \quad (1)$$

B. Scheduling Scheme

We propose the following scheduling scheme:

1) We construct M honey comb topologies such that in each topology there is only one representative node for each hexagonal cell. Each node is connected to a node in the neighbouring cell to form a connected graph.

2) Let $\Delta > 0$ be a network protocol parameter to prevent simultaneous transmission of conflicting nodes. We thus define that two cells are called *interfering neighbors* if there is a point in one cell which is within a distance $(2+\Delta)r_n$ of some point in another cell. It can be argued easily on the lines of Lemma 4.3 of [1] that each cell has no more than c_1 interfering neighbours where c_1 scales as $(1+\Delta^2)$. We use Lemma 4.4 of [1] which proves that if there are no more than c_1 interfering cells, then at most $(1+c_1)$ colors are required to color the graph.
3) Each of the independent graphs formed can be scheduled into $(1+c_1)$ slots and there are M such graphs to be scheduled. Hence, the schedule length is $M(1+c_1)$.

IV. New Bounds On The Capacity Of Wireless Network

We proceed by first explaining the physical interference model which incorporates fading and shadowing.

A. The Physical Interference Model

All nodes transmit with same power P and it is assumed that *there is no retransmission*. Let Υ denote the set containing simultaneously transmitting nodes. Consider a transmission from a node $X_i; i\epsilon\Upsilon$ received by X_j. Let X_i and $X_k; k\epsilon\Upsilon; k \neq i$ be the simultaneously transmitting nodes. Let N_0 be the thermal noise power spectral density and α be the path loss factor. $V(\cdot)$ and $W(\cdot)$ are random variables corresponding to channel gains due to Rayleigh fading and lognormal shadowing respectively. We assume that $V(\cdot)$ and $W(\cdot)$ are i.i.d. random variables with probability density functions (pdfs) [10],

$$f_V(v) = \frac{1}{\sigma_V^2} e^{\frac{-v}{\sigma_V^2}} u(v),\ f_W(w) = \frac{1}{\sqrt{2\pi}\sigma_W} e^{\frac{-w^2}{2\sigma_W^2}}$$

where $u(\cdot)$ is the unit step function. We define Signal to Noise and Interference Ratio ($SINR$) as;

$$SINR = \frac{\frac{P}{|X_i - X_j|^\alpha} V(i,j) 10^{W(i,j)}}{N_0 + \sum_{k,k\neq i} \frac{P}{|X_k - X_j|^\alpha} V(k,j) 10^{W(k,j)}}. \quad (2)$$

(Note that some recent works have suggested the path loss model to denote the received power by $\frac{P}{(1+d)^\alpha}$ instead of $\frac{P}{d^\alpha}$ where P is the power transmitted by node and d is the distance between transmitter and receiver. Such a model bounds the maximum power transmitted by the node. However, since our analysis is asymptotic, we have retained the classical path loss model.) We assume that the fraction of times for which packets are successfully transmitted is equivalent to the condition that on an average $SINR$ remains above certain threshold γ_c (which is determined by the modulation scheme and the desired data rate) or the expectated value of Signal to Noise Interference Ratio ($E(SINR)$) is lower bounded by γ_c.

Lemma 1: Under the physical interference model described, for each successful transmission, $SINR$ is lower bounded by a threshold γ_c if Δ is chosen to be such that

$$(1+\frac{\Delta}{2})^\alpha \geq 6\gamma_c \frac{((\frac{\sqrt{3}}{\sqrt{3}-1})^\alpha + 2 + \frac{1}{\alpha-1} + \frac{1}{\alpha-2})}{3^{\frac{\alpha}{2}}}. \quad (3)$$

Proof: Any two simultaneously transmitting nodes are separated by at least $(2+\Delta)r_n$. The surface is divided into disjoint discs of radius $(1+\frac{\Delta}{2})r_n$. Note that only one disc has the receiving node which is separated from the transmitter by a distance at most r_n and rest of the neighboring discs contribute to interference. The interfering nodes are the centres of these discs. To obtain an upper bound on the interference, we allow the maximum interference on the receiving node, i.e., we now pack the surface S^2 into discs of radius $(1+\frac{\Delta}{2})r_n$ in a hexagonal packing (as in [11]) with the central disc containing the receiver. In this construction, we have $6k$ nodes $(1 \leq k \leq K_0)$, lying at a distance of $2k(1+\frac{\Delta}{2})r_n$ from the transmitter. The minimum distance of each of these nodes from the receiver is $(1+\frac{\Delta}{2})r_n(\sqrt{3}k-1)$.

$$\sum \frac{P}{\mid X_k - X_j \mid^\alpha} \leq \frac{6P}{(\sqrt{3}r_n)^\alpha (1+\frac{\Delta}{2})^\alpha} \Sigma_k \frac{k}{(k-\frac{1}{\sqrt{3}})^\alpha} \quad (4)$$

Since $V(\cdot)$ and $W(\cdot)$ are i.i.d. to each other, we have

$$E(SINR) = \frac{\frac{P}{|X_i - X_j|^\alpha} E(V(i,j)10^{W(i,j)})}{E(N_0 + \sum_{k,k\neq i} \frac{P}{|X_k - X_j|^\alpha} V(k,j)10^{W(k,j)})}. \quad (5)$$

Using Jansen Inequality for convex functions, we obtain

$$\begin{aligned} E(SINR) &\geq \frac{\frac{P}{|X_i - X_j|^\alpha} E(V(i,j)10^{W(i,j)})}{E(N_0 + \sum_{k,k\neq i} \frac{P}{|X_k - X_j|^\alpha} V(k,j)10^{W(k,j)})} \\ &\geq \frac{\frac{P}{|X_i - X_j|^\alpha} E(V(i,j)10^{W(i,j)})}{N_0 + \sum_{k,k\neq i} \frac{P}{|X_k - X_j|^\alpha} E(V(k,j)10^{W(k,j)})}. \quad (6) \end{aligned}$$

It can be easily proved that $E(V(i,j)10^{W(i,j)})$ is convergent and equal to some constant, say σ_0. Thus

$$E(SINR) \geq \frac{\frac{P\sigma_0}{N_0}}{r_n^\alpha + \frac{6\frac{P}{N_0}\sigma_0}{(\sqrt{3})^\alpha(1+\frac{\Delta}{2})^\alpha}((\frac{\sqrt{3}}{\sqrt{3}-1})^\alpha + 2 + \frac{1}{\alpha-1} + \frac{1}{\alpha-2})}. \quad (7)$$

Therefore with $(1+\frac{\Delta}{2})^\alpha$ chosen as in Equation 3 and for large n, RHS becomes $\geq \gamma_c$. ■

B. Packet Drop (Capture) Probability :

For a finite $SINR$, there is always a non-zero probability of packet loss associated with a wireless link. We assume that each transmission (hop) is associated with a finite probability of packet transfer success given by a map $\phi : \beta \longrightarrow \phi(\beta)$, where $\beta = E(SINR)$. For a finite β, $\phi(\beta) < 1$. As $\beta \to \infty$, $\phi(\beta) \to 1$. Also, for $\beta < \gamma_c$, $\phi(\beta) = 0$. Since $E(SINR) \geq \gamma_c$, the packet transfer success probability is lower bounded by $\phi(\gamma_c)$. The power from a transmitter to a receiver is upper bounded by $\frac{P}{(t\rho_n)^\alpha}$ where $0 < t < 1$. Since

$$SINR = \frac{\frac{P}{|X_i - X_j|^\alpha} V(i,j)10^{W(i,j)}}{N_0 + \sum_{k,k\neq i} \frac{P}{|X_k - X_j|^\alpha} V(k,j)10^{W(k,j)}} \quad (8)$$

$$\leq \frac{\frac{P}{|X_i - X_j|^\alpha} V(i,j)10^{W(i,j)}}{\sum \frac{P}{|X_k - X_j|^\alpha} V(k,j)10^{W(k,j)}} \leq \frac{\frac{P}{(t\rho_n)^\alpha} V(i,j)10^{W(i,j)}}{I_{min} V(k,j)10^{W(k,j)}} \quad (9)$$

where I_{min} denotes the minimum interference which is equal to the interference by single farthest node from the receiver.

$$E(SINR) \leq \frac{\frac{P}{(t\rho_n)^\alpha}}{\frac{P}{(r_n+1)^\alpha}} = (\frac{8\rho_n+1}{t\rho_n})^\alpha \sigma_0 = \beta_0 \qquad (10)$$

Hence, the probability of successful packet transfer is upper bounded by a fixed constant $\phi(\beta_0) < 1$.

C. Determining Complexity of Throughput Capacity

Theorem 1: The per node throughput capacity of the network is $\Theta(\frac{1}{n})$.

Proof: Let $\lambda_n(i)$ be the rate in packets/slot at which the source node injects packets into the network. Let p_{ji} denote the probability that packet is received successfully over the j^{th} hop of connection i. However, since all the packets do not reach the destination, the actual end to end throughput of connection i (denoted by $\Lambda_n(i)$) is given by
$\Lambda_n(i) = \lambda_n(i)\Pi_{H_i}\ p_{ji}$.
As already proved, the schedule length $K = 12\sqrt{3}\log n(1 + c_1)$. Each node gets at least one opportunity and maximum of $(1 + c_1)$ opportunities to transmit in K slots, say W bits. Using Equation (1), we have

$$\lambda_n(i)\{\phi(\beta)_{min}\}^{\frac{4L_i}{3\sqrt{3}\rho_n}} \leq \Lambda_n(i) \leq \lambda_n(i)\{\phi(\beta)_{max}\}^{\frac{L_i}{8\rho_n}} \qquad (11)$$

Since L_i and H_i are i.i.d. random variables, the throughput capacity is determined by taking its expectation value. From [8], we know that, for $0 \leq \delta \leq 1$, $E_L[\delta^{L_i}] = \frac{2\pi(1+\delta^{\frac{\sqrt{\pi}}{2}})}{4\pi+(\log\delta)^2}$. Hence, we obtain

$$\frac{\frac{27}{8}\pi\rho_n^2\lambda_n(i)}{(\frac{27}{4}\pi\rho_n^2 + \log^2\phi(\gamma_c))} \leq \Lambda_n(i) \leq \frac{256\pi\rho_n^2\lambda_n(i)}{(256\pi\rho_n^2 + \log^2\phi(\beta_0))}. \qquad (12)$$

Since $\frac{n}{\log n}$ is much larger than $\frac{1}{\log^2\phi(\gamma_c)}$ for large n, we have

$$\frac{9\rho_n^2 W\pi^2}{64\sqrt{3}(1+c_1)\log^2\phi(\gamma_c)\log n} \leq \Lambda_n(i) \leq \frac{64\rho_n^2 W\pi^2}{3\sqrt{3}\log^2\phi(\beta_0)\log n} \qquad (13)$$

or,

$$\frac{3\sqrt{3}W\pi^2}{64(1+c_1)\log^2\phi(\gamma_c)}\frac{1}{n} \leq \Lambda_n(i) \leq \frac{64\sqrt{3}W\pi^2}{9\log^2\phi(\beta_0)}\frac{1}{n}. \qquad (14)$$

Hence $\Lambda_n(i) = \Theta(\frac{1}{n})$. ■

V. Conclusions

We have tightened the constructive upper bound of [8] to derive the complete order of complexity of capacity of the network. The analysis reflects both the packet drop probability as well as fading and shadowing effects. In our analysis, we have kept communication range fixed for simplicity. We can also have a model which considers variable communication range which might help improving the capacity. The possibility of judicious assignment of power in scheduling (variable power assignment) may also add to capacity. Like [9], we can incorporate fading combating strategies in our routing and scheduling scheme which may help in achieving super linear growth in transport capacity. These issues are currently under investigation.

Appendix

Lemma 2: Each hexagonal cell has at least one node with high probability (*whp*). The number of nodes in the cell is bounded by $M = \frac{12\sqrt{3}\log n}{\pi}$, *whp*.

Proof: Let A be the event that a node in the given random network lies in a hexagonal cell and A_n be the event that none of the n nodes lie in the cell. Since the node distribution is an i.i.d. we obtain

$$P(A_n) = (1 - P(A))^n = (1 - \frac{6\sqrt{3}\log n}{\pi n})^n \qquad (15)$$

Using L'opital rule we obtain $P(A_n) \rightarrow 0$ as $n \rightarrow \infty$. Hence probability that at least one node is present in a cell tends to 1 as n tends to infinity. Let $X_k, k\epsilon\{1, 2, 3, \ldots n\}$ be i.i.d. indicator variables denoting the presence of a node in a randomly chosen cell C_k. $X_k\ \epsilon\ [0, 1]$. The probability of a node being present in the cell is $p = E(X_k) = \frac{6\sqrt{3}\log n}{\pi n}$. Hence the number of nodes present $M_i = \sum_1^n X_k$ in a cell C_i is a binomial distribution $B(n;p)$ with $E(M_i) = np$ and $Var(M_i) = np(1-p)$. Consider $X_n = \frac{\sum_1^n X_k}{n}$ and $Var(X_k) = \sigma^2 = \frac{Var(M_i)}{n}$. Now using law of large number for some $\epsilon > 0$

$$P(\mid X_n - p \mid < \epsilon) \geq 1 - \frac{\sigma^2}{\epsilon^2 n} \qquad (16)$$

using $\epsilon = p$

$$P(|M_i - \frac{6\sqrt{3}\log n}{\pi}| < \frac{6\sqrt{3}\log n}{\pi}) \geq 1 - \frac{\pi}{6\sqrt{3}\log n} \qquad (17)$$

or number of nodes in a cell is bounded by $M = \frac{12\sqrt{3}\log n}{\pi}$, *whp*. ■

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